Combinatorial resolutions of multigraded modules and multipersistent homology

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Abstract

Let $R = k[x_1 \ldots x_r]$ and M a multigraded $R-$module. In this work we interpret M as a multipersistent homology module and give a multigraded resolution of it. The construction involves cellular resolutions of monomial ideals and reflects the combinatorial structure of multipersistence homology modules. In the one critical case, a multifiltration is represented by a labelled cellular complex. A multipersistence homology module measures the defect of acyclicity of the associated multigraded cellular chain complex.

1 Introduction

The theory of Persistent Homology is a very recent and active branch of algebraic topology. Although the theory has a wide range of applications varying from data analysis and shape recognition to network theory, we will focus on it's theoretical framework.

Given an increasing sequence of simplicial complexes, parameterized by the natural numbers, persistent homology detects topological features that are in the simplicial complexes for many values of the parameter, for a complete exposition we refer to [6],[7],[9],[10]. Multipersistent homology is a generalization of persistent homology in which the sequence of simplicial complexes is indexed by vectors in $\mathbb{N}^r$, first introduced by Carlsson and Zomorodian in [5], [8]. In short, the problem of multipersistent homology is about calculating the simplicial homologies of the spaces and confronting them using the maps induced in homology by the inclusions. From a dynamical point of view, we want to study how homology evolves along the sequence of spaces, restricting to the case when the process is stationary.

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The study of multipersistent homology is carried in [5] and [8] through multigraded modules over the polynomial ring $R := k[x_1 \dots x_r]$ called multipersistent modules.

In this work we highlight how, in analogy to simplicial homology, multipersistent homology modules are the homologies of the chain complex of $n-$chain modules. The module of $n-$ chains can be composed as a direct sum of monomial ideals.

This characterization, provides a presentation of multipersistent homology modules by generators and relations that reflects the topological and combinatorial nature of the problem. In a special case, called one-critical in [8], the chain complex is a chain complex of free $R-$modules. This free chain complex is isomorphic to a cellular resolution if it is acyclic. In most cases anyway the complex is not acyclic and multipersistent homology modules measure the defect of acyclicity.

Using our presentation and tools from homological algebra we then build a free multigraded resolution for multipersistent homology modules. As a constructive theorem [5] states that every multigraded R-module can be realized as a multipersistent homology module, this resolution applies to all multigraded R-modules.

The second section of this article is dedicated to notation and background information. The central part of the article is the third section in which we give a presentation of multipersistent homology modules and build a standard free resolution for such modules, other resolutions for multigraded R-modules with combinatorial meaning can be found in [3], [17]. Cellular resolutions of monomial ideals [13],[1],[2] are used in two ways through the article: firstly they are used in the construction of our resolution of multipersitent homology modules; secondly we show how, in the one critical case, multipersistent homology modules measure the defect of aciclicity of cellular chain complexes constructed for cellular resolutions.

2 Background

2.1 Multigraded modules

In this section we recall some notions from topology, homological and commutative algebra that will be used in the article.

Throughout the article k will denote an arbitrary field of characteristic 0 and Vct_k the category of k-vector spaces and linear maps. For a commutative finitely generated $k-$algebra A, we denote with Mod_A the category of $A-$modules and $A-$module homomorphisms. For us A will be the field k or the algebra of polynomials in r variables $R := k[x_1 \dots x_r]$.

The R-modules we will encounter are multigraded by $\mathbb{N}^r$. We consider $\mathbb{N}^r$ as a partially ordered set by setting $v \preceq w$ for $v = (v_1, \dots, v_r)$ and $w = (w_1, \dots, w_r)$ if $v_i \leq w_i$ for all $i = 1, \dots, r$.

Given $v = (v_1 \dots v_r) \in \mathbb{N}^r$, we denote the monomial $x_1^{v_1} \cdots x_r^{v_r}$ with $\underline{x}^v$. The polynomial algebra R has a multigraded decomposition as

$$R = \bigoplus_{v \in \mathbb{N}^r} k \cdot \underline{x}^v. \tag{2.1}$$

Definition 2.1. *A multigraded module over R is a module M with a vector space decomposition $M = \bigoplus_{v \in \mathbb{N}^r} M_v$ such that $R_w \cdot M_v \subseteq M_{w+v}$ for all v, $w \in \mathbb{N}^r$. A homomorphism of modules that preserves the multigrading is a homomorphism of multigraded modules. Multigraded modules and homomorphisms determine a category.*

To a family of vector spaces $\{M_v\}_{v \in \mathbb{N}^r}$ and linear maps $\varphi_{v,w} : M_v \to M_w$ for all $v \preceq w$, such that $\varphi_{v,w} = \varphi_{z,w} \cdot \varphi_{v,z}$, for all $v \preceq z \preceq w$, we can associate the vector space $M = \bigoplus_v M_v$ with R-module action

$$\begin{array}{rcll} x_i : M_v & \longrightarrow & M_{v+e_i} & \qquad 0 \leq i \leq r \\ m & \longrightarrow & \varphi_{v,v+e_i}(m) & \end{array}$$

where e_i is the vector with $i-$th entry equal to 1 and 0 elsewhere. This gives, as it is easy to check, an equivalence of categories between functors from $\mathbb{N}^r$ to Vct_k and multigraded R-modules.

In particular a $\mathbb{N}^2-$graded module is a lattice of $k-$vector spaces and commuting linear maps. The correspondence between families of vector spaces and multigraded modules has been studied also in other contexts, see [16].

2.2 Homology

We write Ch_A to denote the category of chain complexes of A-modules i.e. $(C, \partial) = (C_n, \partial_n)_{n \in \mathbf{Z}}$ with $C_n \in Mod_A$ and $\partial_n \partial_{n+1} = 0$, and chain maps $\alpha = \{\alpha_n\} : (C_n, \partial_n) \to (D_n, \delta_n)$ i.e. $\alpha_n : C_n \to D_n$ are $A-$modules morphisms such that $\delta_n \alpha_n = \alpha_{n-1} \partial_n$.

Fixed a chain complex $C = (C_n, \partial_n)_{n \in \mathbf{Z}}$, the kernel of ∂_n is the module of n-cycles of C, denoted $Z_n = Z_n(C)$; the image of $\partial_{n+1} : C_{n+1} \to C_n$ is the module of n-boundaries of C, denoted $B_n = B_n(C)$.

Definition 2.2. *The n-th homology module of C is the $A-$module*

$$H_n(C) = Z_n(C)/B_n(C). \tag{2.2}$$

The assignment $C \to H_n(C)$ induces a covariant functor from Ch_A to Mod_A, see [18].

A chain complex C is said exact if $H_n(C) = 0$ for all n.

Definition 2.3. *Let M be a R-module. A multigraded free resolution of M is a chain complex (C, ∂) of multigraded free R-modules with $C_i = 0$ for $i < 0$, together with a homomorphism $\epsilon : C_0 \to M$ so that the chain complex*

$$\cdots \to C_2 \stackrel{\partial_2}{\to} C_1 \stackrel{\partial_1}{\to} C_0 \stackrel{\epsilon}{\to} M \to 0$$

is exact. The resolution of M is often denoted as $C \to M$.

When multigraded free resolutions in the category Mod_R will be used we denote by $R(-v)$ the free R-module with one generator in multidegree $v \in \mathbb{N}^r$.

We will now briefly introduce simplicial homology with coefficients in k, a general treatment of the subject can be found in [15].

A simplicial complex is a non empty family $\mathcal{K}$ of finite subsets, called faces, of a universal set; such that if $\sigma \in \mathcal{K}$ and $\sigma' \subset \sigma$, then $\sigma' \in \mathcal{K}$. The faces of cardinality one are called vertices. We assume that the vertex set is finite and totally ordered. A face of $n+1$ vertices is called $n-$face and denoted by $[p_0, \ldots, p_n]$. The dimension of a simplicial complex is the highest dimension of the faces in the complex. A simplicial map is a map between simplicial complexes with the property that the image of a vertex is a vertex and the image of a $n-$face is face of dimension $\leq n$. Simplicial complexes and simplicial maps determine a category that we denote by SC. Fixed a simplicial complex $\mathcal{K}$ of dimension d, we denote by $\mathcal{K}_n$ the set of n-faces in $\mathcal{K}$, for $n = 0, \ldots d$. The set of $n-$faces and $(n-1)-$faces are linked by $n+1$ set maps

$$\begin{array}{rcll} d_i : \mathcal{K}_n & \longrightarrow & \mathcal{K}_{n-1} & \quad 0 \leq i \leq n \\ [p_0, \ldots, p_n] & \to & [p_0, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n]. & \end{array}$$

The vector space $C_n(\mathcal{K})$ on the set $\mathcal{K}_n$ is called vector space of $n-$chains. The set maps d_i yield linear maps $C_n(\mathcal{K}) \to C_{n-1}(\mathcal{K})$ which we also call d_i. This data defines a functor $C_n : SC \to Vct_k$.

Definition 2.4. *The simplicial chain complex of $\mathcal{K}$ with coefficients in k is the chain complex*

$$C_{\mathcal{K}} : \qquad 0 \to C_d \stackrel{\partial_d}{\to} C_{d-1} \stackrel{\partial_{d-1}}{\to} \cdots \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0. \tag{2.3}$$

with the differential operator $\partial_j = \sum_{i=0}^{n} (-1)^i d_i$ for $j = 1 \ldots d$.

The assignment $\mathcal{K} \to C_{\mathcal{K}}$ induces a functor from SC to Ch_k.

Definition 2.5. *The $n-th$ homology group of the simplicial complex $\mathcal{K}$ is the $n-th$ homology of the simplicial chain complex $C_{\mathcal{K}}$.*

In terms of categories, simplicial homology is the restriction of the functor H_n to simplicial chain complexes.

2.3 Multipersistence Homology Modules

In this article we will follow the way traced by [5], [8] to study multipersistent homology through multigraded modules over the polynomial ring.

As in [5], we call a topological space X *multifiltered* if we are given a family of subspaces $\{X_v\}_v = \{X_v\}_{v\in\mathbb{N}^r}$, so that $X_v \subseteq X_w$ whenever $v \preceq w$. The family $\{X_v\}_v$ is called a *multifiltration*.

From now on we denote with X a multifiltered simplicial complex and with $\{X_v\}_v$ a multifitration of it.

Consider the functor of n chains $C_n : SC \to Vct_k$ applied to the multifiltration $\{X_v\}_v$, we have a family of vector spaces $\{C_n(X_v)\}_v$ and linear inclusions between them $\{C_n(X_v \hookrightarrow X_w)\}_{v\preceq w}$. Sometimes, for the sake of simplicity, we shorten $C_n(X_v)$ by $C_n(v)$.

These determine an $R-$module $C_n := \bigoplus_v C_n(v)$ with module action

$$\underline{x}^w := C_n(X_v \hookrightarrow X_{v+w}) : C_n(v) \to C_n(v+w),$$

for $v, w \in \mathbb{N}^r$.

Definition 2.6. *The $n-$chain module of the multifiltration $\{X_v\}_v$ is the multigraded $R-$module C_n.*

Consider now the $n-$homology functor $H_n : SC \to Vct_k$ applied to the multifiltration $\{X_v\}_v$, we have a family of vector spaces $\{H_n(X_v)\}_v$ and linear maps (not necessarily inclusions) between them $\{H_n(X_v \hookrightarrow X_w)\}_{v\preceq w}$. Again, we will write $H_n(v)$ for $H_n(X_v)$ when X is clear from the context.

These determine an $R-$module $H_n := \bigoplus_v H_n(v)$ with module action

$$\underline{x}^w := H_n(X_v \hookrightarrow X_{v+w}) : H_n(v) \to H_n(v+w),$$

for $v, w \in \mathbb{N}^r$.

Definition 2.7. *The $n-$multipersistent homology module is the multigraded $R-$module H_n.*

Exploiting this module structure, we will investigate the structure of multipersistent homology modules and their link to $n-$chain modules in the context of combinatorial commutative algebra.

3 Main results

The first step in our analysis of multipersistence homology modules consists in studying the structure of $n-$chain modules.

Definition 3.1. *A multifiltration $\{X_v\}_v$ is stationary if there is $v' \in \mathbb{N}^r$ such that for all $v \in \mathbb{N}^r$ with $v_i \geq v'_i$ for some i, one has $X_{v+ke_i} = X_v$, for all $k \in \mathbb{N}$.*

Being the complex X finite, any multifiltration of X is stationary. Let's consider $\mathbf{B}_n(v)$ the basis of $C_n(X_v)$ with elements corresponding to the $n-$faces in X_v. The set of bases $\{\mathbf{B}_n(v)\}_{v\in\mathbb{N}^r}$ is such that

$$\underline{x}^w\mathbf{B}_n(v) \subseteq \mathbf{B}_n(v+w), \tag{3.1}$$

for all v, $w \in \mathbb{N}^r$.

Definition 3.2. *Let $\sigma \in \mathbf{B}_n(v')$ be a basis element corresponding to a $n-$face in X. A critical coordinate for σ, is a minimal $v \in \mathbb{N}^r$ such that there is $\tau \in \mathbf{B}_n(v)$ with $\underline{x}^{v'-v}\tau = \sigma$. The element τ is a fundamental element associated to σ.*

In general, the critical coordinate and fundamental element for $\sigma \in \mathbf{B}_n(v')$ are not unique. We denote by $\mathcal{F}_\sigma$ the set of fundamental elements associated to $\sigma \in \mathbf{B}_n(v')$ and $\mathcal{F} = \bigcup_{\sigma\in\mathbf{B}_n(v')}\mathcal{F}_\sigma$.

As the vector spaces $C_n(v)$ are all finite dimensional for all $v \in \mathbb{N}^r$ and the multifiltration is stationary, the module C_n is finitely generated. By construction the set $\mathcal{F}$ minimally generates C_n. We denote by $deg\,\tau$ the degree of the generator $\tau \in \mathcal{F}$, i.e if $\tau \in \mathbf{B}_n(v)$ then $deg\,\tau = v$.

Lemma 3.3. *The first syzygy module of C_n is minimally generated by binomials of the form*

$$\underline{x}^{c-\deg a}a - \underline{x}^{c-\deg b}b,$$

where $a, b \in \mathcal{F}_\sigma$, $c \in \mathbb{N}^r$ with $c_i = \max((deg\,a)_i, (deg\,b)_i)$ for $i : 1 \ldots r$, and $\sigma \in \mathbf{B}_n(v')$.

Proof. As C_n is a multigraded module, all the relations in C_n are homogeneous and a relation in degree v is of the form

$$\sum_{m\in\mathcal{F}} \lambda_m x^{v-\deg m}m$$

with $\lambda_m \in k$.

Furthermore, one can easily check that given $a, b \in \mathcal{F}$, for c defined as in the statement, if $c \preceq v$, it holds $\underline{x}^{v-\deg a}a = \underline{x}^{v-\deg b}b$ if and only if $a, b \in \mathcal{F}_\sigma$ for some $\sigma \in \mathbf{B}_n(v')$. Therefore

$$\sum_{m\in\mathcal{F}} \lambda_m x^{v-\deg m}m = \sum_{\sigma\in\mathbf{B}_n(v')} (\sum_{m\in\mathcal{F}_\sigma} \lambda_m)m_\sigma$$

where $m_\sigma \in B_n(v)$ is the unique basis elements corresponding to $\sigma \in \mathbf{B}_n(v')$. Hence $\sum_{m\in\mathcal{F}_\sigma} \lambda_m = 0$ and the result follows. ☐

Theorem 3.4. *The module of $n-$chains can be decomposed as a direct sum of monomial ideals in the following way:*

$$C_n \cong \bigoplus_{\sigma \in \mathbf{B}_n(v')} < (\underline{x}^{deg\, a})_{a \in \mathcal{F}_\sigma} > .$$

where $< (\underline{x}^{deg\, a})_{a\in\mathcal{F}_\sigma} >$*is the* $R-$*ideal generated by the monomials* $\underline{x}^{deg\, a}$ *for* $a \in \mathcal{F}_\sigma$.

Proof. Let $C_{n,\sigma}$ be the submodule of C_n generated by the set $\mathcal{F}_\sigma$. First we prove the decomposition $C_n = \bigoplus_{\sigma\in\mathbf{B}_n(v')} C_{n,\sigma}$. Since C_n is generated by $\mathcal{F} = \cup\mathcal{F}_\sigma$ it is clear that the canonical map $\bigoplus_\sigma C_{n,\sigma} \to C_n$ is onto. The claim follows by the very proof of previous Lemma 3.3. We show now that the submodules $C_{n,\sigma}$ are isomorphic to monomial ideals.

The following defines an injective homomorphism of $R-$modules,

$$\begin{array}{rcl} C_{n,\sigma} & \to & R \\ \mathcal{F}_\sigma \ni a & \to & \underline{x}^{\deg a}. \end{array}$$

Consider the free $R-$module F_σ generated by $\mathcal{F}_\sigma$. The assignment $a \to \underline{x}^{\deg a}$, for all $a \in \mathcal{F}_\sigma$ defines a homomorphism of $R-$modules $\varphi : F_\sigma \to R$. We denote by I_σ the kernel of the natural homomorphism $F_\sigma \to C_{n,\sigma}$. By the previous Lemma 3.3 it follows that $I_\sigma \subset \ker\varphi$ and it is easy to see that actually they coincide. This gives an isomorphism $C_{n,\sigma} \cong \varphi(F_\sigma)$ and the latter is clearly a monomial ideal. □

Let $d = \dim X$, the $R-$modules C_n fit in the chain complex of $R-$modules

$$C: \qquad 0 \to C_d \xrightarrow{\partial_d} \ldots \to C_1 \xrightarrow{\partial_1} C_0 \to 0. \tag{3.2}$$

which is the direct sum of the simplicial chain complexes C_{X_v}, see Def.2.4. Indeed, if $a \in \mathcal{B}_n(v)$ corresponds to $\sigma \in X_v \subset X$, then $\underline{x}^{e_j}a$ corresponds to the same face. Thus $\underline{x}^{e_j} \cdot \partial_n = \partial_n \cdot \underline{x}^{e_j}$. By construction $\partial_n(C_n(v)) \subset C_{n-1}(v)$ and $\underline{x}^{e_j}(C_n(v)) \subset C_n(v + e_j)$. Thus C is doubly graded

$$C = \bigoplus_{n=0}^{d} C_n = \bigoplus_{n=0}^{d}(\bigoplus_{v\in\mathbb{N}^r} C_n(v)) = \bigoplus_{v\in\mathbb{N}^r}(\bigoplus_{n=0}^{d} C_n(v)) = \bigoplus_{v\in\mathbb{N}^r} C(v). \tag{3.3}$$

Fixing n we have C_n, the n-th chain module of C; fixing v we have the simplicial chain complex $C(v)$ of X_v.

This gives, in the obvious way, boundaries and cycles modules, $B_n(C) = \bigoplus_v B_n(v)$ and $Z_n(C) = \bigoplus_v Z_n(v)$, which turn out to be multigraded finitely generated $R-$modules as C_n.

As observed also in [8], multipersistent homology modules are the homologies of a chain complex.

Proposition 3.5. *Multipersistent homology modules are the homology modules of* C.

Proof. It is enough to recall that homology functors commute with direct sums, i.e.

$$\bigoplus_v H_n(X_v) = H_n(\bigoplus_v X_v) = H_n(C)$$

□

In analogy with the standard case, in which simplicial homology modules are calculated from the simplicial chain complex, we use C to calculate multipersistence homology modules. In this sense multipersistence homology modules can be thought as the simplicial homology of a multifiltration.

Opposite to the chain modules, multipersistent homology modules are not monomial, in general. Every multigraded R-module can be realized as a multipersistent homology module, as claimed in [5]. The next step is to exploit our presentation to build a resolution for multipersistent homology modules, from cellular resolutions of monomial ideals.

3.1 Combinatorial Resolution

In this subsection we construct a free resolution of $H_n(C)$, in the category of multigraded R-modules, in terms of resolutions of C_n and $Z_n(C)$. This construction exploits cellular resolutions of monomial ideals.

We need to introduce a new actor on the scene.

Definition 3.6. *If $\alpha : F \to G$ is a map between the chain complexes (F, φ) and (G, η), the mapping cone $M(\alpha)$ of α is the chain complex with components $M(\alpha)_j = F_{j-1} \oplus G_j$ and differentials $\partial_j^\alpha : M_j(\alpha) \to M_{j-1}(\alpha)$ given by the matrices*

$$\partial_j^\alpha := \begin{pmatrix} -\varphi_{j-1} & 0 \\ \alpha_{j-1} & \eta_j \end{pmatrix}$$

The map ∂_n is morphism of multigraded $R-$modules for all n. Therefore $Z_n(C)$ and $B_n(C)$ are multigraded $R-$submodules of C_n and the natural inclusion $i^n : Z_n(C) \to C_n$ is a homomorphism of multigraded $R-$modules. Let us then consider the short exact sequence of R-modules

$$0 \to Z_n(C) \xrightarrow{i^n} C_n \xrightarrow{\partial_n} B_{n-1}(C) \to 0. \tag{3.4}$$

This can be lifted to a chain map between resolutions of $Z_n(C)$ and C_n using the Comparison Theorem.

Theorem 3.7 (Comparison Theorem [18])**.** *Let M and N be R-modules, $P \xrightarrow{\epsilon} M$ be a projective resolution of M and $f' : M \to N$ a morphism of R-modules. Then for every resolution $Q \xrightarrow{\pi} N$ of N there is a chain map $f : P \to Q$ lifting f' in the sense that*

$$\pi \circ f_0 = f' \circ \epsilon. \tag{3.5}$$

where $f_0 : P_0 \to Q_0$. The chain map is unique up to chain homotopy equivalence.

We have seen that C_n is a direct sum of monomial ideals. Monomial ideals admit cellular resolutions, for explanations we refer to [13], [1], [2], [14]. Cellular resolutions are constructed in a completely combinatorial way from a labelled cellular complex associated to the monomial ideal. We then consider cellular resolutions for the components of C_n and their direct sum (P^n, φ^n) is resolution of C_n.

The module $Z_n(C)$ of $n-$cycles is not a direct sum of monomial ideals because a set of bases of the homogeneous components with the property 3.1 cannot be defined. Let (Q^n, η^n) be any resolution of $Z_n(C)$.

We denote by $I^n : P^n \to Q^n$ the chain map induced by i^n (see (3.4)) on the resolutions.

Lemma 3.8. *The mapping cone $M(I^n)$ with differential*

$$\partial_j^{M(I^n)} = \begin{pmatrix} -\varphi_{j-1}^n & 0 \\ I_{j-1}^n & \eta_j^n \end{pmatrix}$$

is a free resolution of $B_{n-1}(C)$.

Proof. For a complex C, we denote by $C[-n]$ the shift of the chain complex such that $C[-n]_k = C_{k-n}$.

The short exact sequence

$$0 \to Q^n \to M(I^n) \to P^n[-1] \to 0 \tag{3.6}$$

induces a long exact sequence in homology with connecting homomorphism the map induced in homology by I^n,

$$\ldots \to H_k(Q^n) \to H_k(M(I^n)) \to H_{k-1}(P^n) \stackrel{H_{k-1}(I^n)}{\to} H_{k-1}(Q^n) \ldots . \tag{3.7}$$

From 3.7 we deduce that $H_k(M(I^n)) = 0$ for $k \geq 1$ because the chain complexes P^n and Q^n are acyclic and that the following sequence is exact

$$0 \to H_1(M(I^n)) \stackrel{\nu}{\to} Z_n(C) \stackrel{H_0(I)}{\to} C_n \to H_0(M(I^n)) \to 0.$$

Because of 3.5, $H_0(I) = i^n$ hence $Im\,\nu = \ker \nu = 0$ and also $H_1(M(I)) = 0$. □

We can use this resolution to construct a free resolution of $H_n(C)$.

Proposition 3.9. *The chain complex*

$$\ldots \to P_{j-2}^{n+1} \oplus Q_{j-1}^{n+1} \oplus P_j^n \stackrel{\delta_j}{\to} \ldots \to Q_0^{n+1} \oplus P_1^n \stackrel{\delta_1}{\to} P_0^n \tag{3.8}$$

with differential

$$\delta_j(x, y, z) = (\varphi_{j-2}^{n+1}(x), -I_{j-2}^{n+1}(x) - \eta_{j-1}^{n+1}(y), L_{j-1}(x, y) + \varphi_j^n(z)) \tag{3.9}$$

is a finite free multigraded resolution of $H_n(C)$.

Proof. Consider the short exact sequence of R-modules

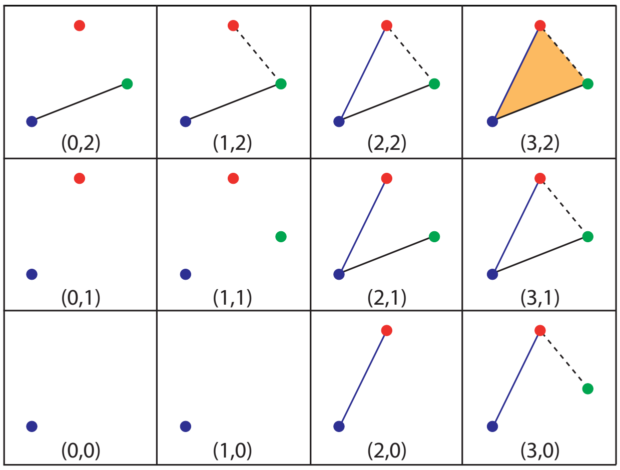
$$0 \to B_n(C) \xrightarrow{l} Z_n(C) \to H_n(C) \to 0 \tag{3.10}$$

and resolutions $(M(I^{n+1}), \partial^{M(I^{n+1})})$ of $B_n(C)$ and (P^n, φ^n) of $Z_n(C)$.

Repeating the mapping cone argument, l induces a chain map $L : M(I^{n+1}) \to P^n$. The mapping cone $M(L)$ is the desired free resolution for $H_n(C)$. $\square$

3.2 Example

We will now write the chain complex C and the combinatorial resolution for the example considered in [5].



In this example:

$$\begin{aligned} C_0 &\simeq < 1 > \oplus < xy, x^3, y^2 > \oplus < y, x^2 >, \\ C_1 &\simeq < x^2 > \oplus < y^2, x^2 y > \oplus < xy^2, x^3 >, \\ C_2 &\simeq < x^3 y^2 > . \end{aligned}$$

The free resolution for $H_0(C)$ is

$$0 \to R(-3,-1) \oplus R(-1,-2)^2 \oplus R(-2,-1)^2 \oplus R(-3,0) \oplus R(0,-2) \oplus R(-2,0) \to$$
$$\to R(-3,0) \oplus R(0,-2) \oplus R(-1,-1) \oplus R(-2,0) \oplus R(0,-1) \oplus R(0,0) \to 0.$$

The free resolution for $H_1(C)$ is

$$0 \to R(-3,-2)^2 \to R(-2,-2) \oplus R(-3,-1) \to 0.$$

Remark 3.10. *The presentation matrix δ_1 shows how cycles* are identified *and when they become boundaries so it can be used to detect persistent features in the sequence of simplicial complexes.*

3.3 One-critical case

For a class of multifiltrations the algebraic setting is particularly simple and the chain complex C has an explicit combinatorial meaning.

Definition 3.11. *Let $\{\mathbf{B}_n(v)\}_v$ be canonical. If an element $\sigma \in \mathbf{B}_n(v')$ has a unique critical coordinate, then it is called one critical. If every $\mathbf{B}_n(v')$ is one critical then M called one critical too.*

The simplification in the one-critical case relies in the following proposition

Proposition 3.12. *The module M is one-critical if and only if M is a free R-module.*

Proof. A module M is one-critical if and only if $\{\mathcal{F}_\sigma\}$ is a one element set for all $\sigma \in \mathbf{B}_n(v')$. If we denote by m_σ the unique fundamental element in $\{\mathcal{F}_\sigma\}$, then we have $M_\sigma \simeq R(-\deg m_\sigma)$. □

Definition 3.13. *We say that a multifiltered complex X is one critical if its chain modules C_n are one critical for all n.*

This definition is equivalent to the definition of multifiltered complex given in [8].

Definition 3.14. *A labelled simplicial complex is a simplicial complex X with a function $\Updownarrow : X \to \mathbb{N}^r$.*

In the one critical case the multifiltration can be studied by means of a single labelled simplicial complex.

Definition 3.15. *To a one-critical filtration $\{X_v\}_v$ we associate the simplicial complex X, with faces labelled by their critical coordinate. We denote X labelled as $\tilde{X}$ to distinguish it from the original X.*

Remark 3.16. *This construction is not possible in the general case because the critical coordinate of a face is not in general unique.*

In the example
the associated labelled simplicial complex is represented in Fig.2.

Following the construction in [2] for cellular resolutions, we now associate to $\tilde{X}$ the multigraded complex of free R-modules:

$$F_{\tilde{X}} : 0 \to F_d \xrightarrow{\delta_d} \dots F_n \xrightarrow{\delta_n} \dots \xrightarrow{\delta_1} F_0 \to 0 \tag{3.11}$$

Where d is the dimension of X, F_n is the multigraded free R-module generated by the n-faces in $\tilde{X}$, the multigrading is given by the labels.

The differential δ_n acts on a n-face a labelled by $\underline{x}^{m(a)}$ as

$$\delta_n(a) = \sum_{i=0}^{n} (-1)^i \frac{\underline{x}^{m(a)}}{\underline{x}^{m(d_i(a))}} d_i(a). \tag{3.12}$$

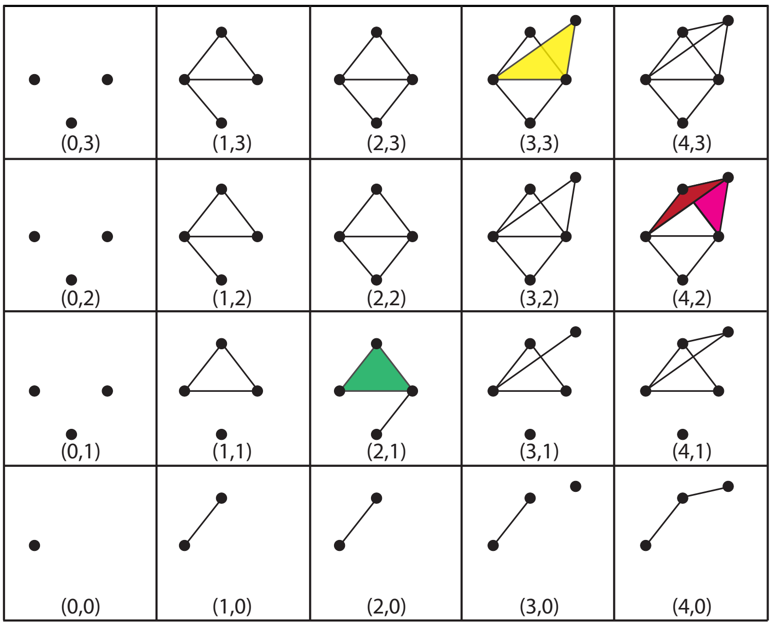


Figure 1: A one critical multifiltration

Remark 3.17. *The labelled complex $\tilde{X}$ is more general than the ones built for cellular resolutions in [2] because the label of a face is not necessarily the lowest common multiple of the labels of its facets.*

The observation that a one-critical filtration is equivalent to a labelled simplicial complex is made more precise in the following proposition.

Proposition 3.18. *If $\{X_v\}_v$ is a one-critical filtration, the chain complex C is equal to $F_{\tilde{X}}$.*

Proof. By construction $C_n = F_n$ as $R-$ modules, for $n : 0 \dots d$. The differential in $F_{\tilde{X}}$ is also the same as the one in C but expresses a coface of the face a as a multiple of the corresponding fundamental element. $\square$

Remark 3.19. *Chain complexes built from labelled cellular complexes are resolutions of monomial ideals under conditions of acyclicity of $\tilde{X}$ [13]. In our case C is certainly not acyclic and multipersistent modules measure this defect.*

In our example the chain complex C is

$$0 \to S(-3,-3) \bigoplus S(-4,-2) \bigoplus S(-2,-1) \xrightarrow{\delta_2} S(-3,-2) \bigoplus S(-3,-1) \bigoplus S(-4,0) \bigoplus S(-1,-2).$$

$$S(-2,-1) \bigoplus S(-1,-1)^2 \bigoplus S(-1,0) \xrightarrow{\delta_1} S(0,-1)^2 \bigoplus S(-3,0) \bigoplus S(-1,0) \bigoplus S(0,0) \to 0$$

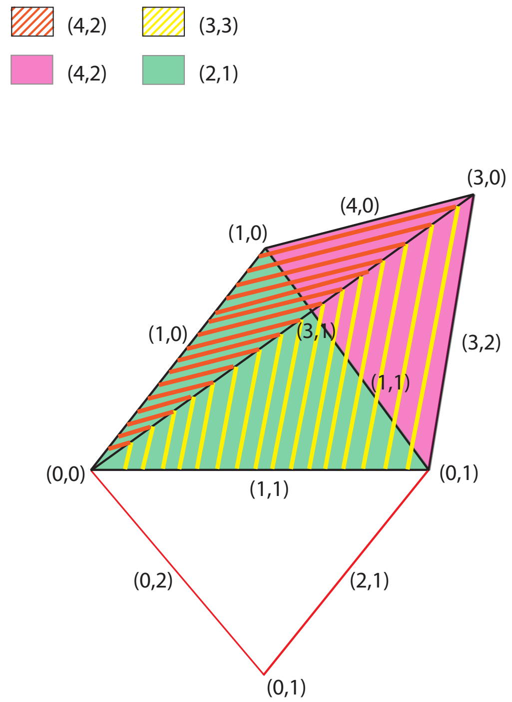


Figure 2: Labeled simplicial complex associated to the filtration in Figure 1

with differentials

$$\delta_1 = \begin{pmatrix} x & xy & 0 & xy^2 & 0 & x^3y & 0 & 0 \\ 0 & 0 & 0 & -xy & x^2 & 0 & 0 & 0 \\ 0 & -x & -x & 0 & -x^2 & 0 & -x^3y & 0 \\ -1 & 0 & y & 0 & 0 & 0 & 0 & x^3 \\ 0 & 0 & 0 & 0 & 0 & -y & y^2 & -x \end{pmatrix} \quad \text{and} \quad \delta_2 = \begin{pmatrix} xy & 0 & x^3y^2 & 0 \\ -x & -x^2y^2 & 0 & -x^3y \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & y^2 & -xy & xy \\ 0 & y & 0 & x \\ 0 & 0 & y^2 & 0 \end{pmatrix}$$

and homology modules

$$H_0 \simeq \operatorname{coker} \begin{pmatrix} -xy & xy^2 & -x^2y & x^3y & x^4 & x^3y^2 \\ 0 & xy & x^2 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y & -x & -y^2 \end{pmatrix} \quad \text{and} \quad H_1 \simeq \operatorname{coker} \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -y & -x & 0 \\ 0 & 0 & 0 & y \end{pmatrix}$$

and the resolution of $H_1(C)$ is

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